MOTION OF A CHARGED GAS FILLING A CYLINDRICAL CAVITY WITH A CROSS SECTION IN THE FORM OF AN ARBITRARY ELLIPSE

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Consider the motion of a one-component charged gas in an external magnetic field, this gas initially filling a cylindrical cavity and having an elliptical cross section. The steady magnetic field lies along the axis of the cylinder, while the induced magnetic field is neglected, since the nonrelativistic case is envisaged, and the currents are small. The motion is essentially two-dimensional, i.e., is dependent on the coordinates (x, y, t). The solution cannot be found in general form, but the investigation can be carried to completion in one case of practical importance. We seek a solution describing the uniform relative deformation of the gas in general form.

Such solutions have been derived in ordinary gasdynamics [1, 2], in magnetohydrodynamics [3], and in gasdynamics in the presence of a gravitational field [4].

The following system of equations describes the motion:

$$m\frac{d\mathbf{u}}{dt} = q\,\mathrm{grad}\,\varphi - \frac{q}{c}\mathbf{u}\times\mathbf{H}_0,$$

$$\frac{\partial n}{\partial t} + \mathrm{div}\,n\mathbf{u} = 0, \qquad \Delta\varphi = 2\pi q n, \tag{1}$$

in which m is the mass of a particle, q is charge, c is the velocity of light, n is density, φ is the electrical potential, and $\mathbf{H}_0$ is the steady magnetic field acting along the Oz-axis.

The gas has a constant density n_0 at t = 0 together with a velocity distribution linear in the coordinates, while the cross section is described by $x^2/a^2 + y^2/b^2 = 1$. This shape of cross section gives φ a quadratic form, and this means that the dependence of the velocity on the coordinates remains linear for $t \neq 0$, with the result that n is uniform over the ellipse at any given t, but the size of the ellipse changes, as does its orientation in space.

We transform (1) by introducing the dimensionless variables

$$n = n_0 N, \qquad u = \sqrt{2\pi q^2 n_0/m}\,aU, \qquad \varphi = 2\pi q n_0 a^2\Phi,$$
$$t = \frac{\tau}{\sqrt{2\pi q^2 n_0/m}}, \qquad x = a\xi, \qquad y = a\eta. \tag{2}$$

In Lagrange variables (ξ_0, η_0, τ) we get

$$\frac{\partial U}{\partial\tau} = \left(\frac{\partial\Phi}{\partial\xi}\right)_{\xi_0,\eta_0} - \sigma V, \qquad \frac{\partial\xi}{\partial\tau} = U\left(\sigma = \frac{qH_0}{mc}\sqrt{\frac{2\pi q^2 n_0}{m}}\right),$$

$$\frac{\partial V}{\partial\tau} = \left(\frac{\partial\Phi}{\partial\eta}\right)_{\xi_0,\eta_0} + \sigma U, \qquad \frac{\partial\eta}{\partial\tau} = V,$$

$$N\frac{\partial(\xi,\eta)}{\partial(\xi_0,\eta_0)} = 1, \qquad \Delta\Phi = N. \tag{3}$$

Here $(\ )_{\xi_0,\eta_0}$ means that the force is written in terms of ξ_0 and η_0. We seek the solution in the form

$$\xi = \xi_0\mu_1(\tau) + \eta_0\psi_1(\tau), \qquad \eta = \xi_0\psi_2(\tau) + \eta_0\mu_2(\tau). \tag{4}$$

The density N is given by

$$N^{-1} = \mu_1\mu_2 - \psi_1\psi_2 = D. \tag{5}$$

Consider how the cross section varies (Fig. 1). The equation of the ellipse at $\tau = 0$ is

$$\xi_*^2 + \beta^{-2}\eta_*^2 = 1 \qquad (\beta = b/a)$$

The axes of the ellipse subsequently vary, as does the orientation. The equation for any t relative to the original position of the axes is

$$(\xi\mu_2 - \eta\psi_1)^2 D^{-2} + \beta^{-2}(\eta\mu_1 - \xi\psi_2)^2 D^{-2} = 1. \tag{6}$$

The boundary remains elliptical, and the equation applicable in a coordinate system linked to the principal axes is

$$\xi_1^2/a_1^2 + \eta_1^2/b_1^2 = 1.$$

Here a_1 and b_1 are the semiaxes in the moving coordinate system, which is turned through θ relative to the initial system; a_1, b_1, and θ

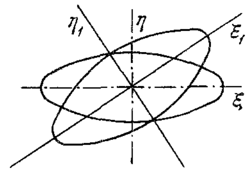

Fig. 1

are given by standard formulas in analytic geometry. The potential in the moving coordinate system is

$$\Phi(\xi_1, \eta_1, \tau) = \Phi_0 + \frac{a_1 b_1 N}{2(a_1+b_1)}\left(\frac{\xi_1^2}{a_1} + \frac{\eta_1^2}{b_1}\right). \tag{7}$$

We substitute (7) into the equations of motion and use the formulas for passing from the moving coordinate system to the fixed one to get

$$\mu_1'' = \frac{a_1b_1N}{2(a_1+b_1)}\left[\mu_1\left(\frac{1}{a_1}+\frac{1}{b_1}\right) + \left(\frac{1}{a_1}-\frac{1}{b_1}\right)(\mu_1\cos 2\theta + \psi_2\sin 2\theta)\right] - \sigma\psi_2',$$

$$\mu_2'' = \frac{a_1b_1N}{2(a_1+b_1)}\left[\mu_2\left(\frac{1}{a_1}+\frac{1}{b_1}\right) + \left(\frac{1}{a_1}-\frac{1}{b_1}\right)(\psi_1\sin 2\theta - \mu_2\cos 2\theta)\right] + \sigma\psi_1',$$

$$\psi_1'' = \frac{a_1b_1N}{2(a_1+b_1)}\left[\psi_1\left(\frac{1}{a_1}+\frac{1}{b_1}\right) + \left(\frac{1}{a_1}-\frac{1}{b_1}\right)(\mu_2\sin 2\theta + \psi_1\cos 2\theta)\right] - \sigma\mu_2',$$

$$\psi_2'' = \frac{a_1b_1N}{2(a_1+b_1)}\left[\psi_2\left(\frac{1}{a_1}+\frac{1}{b_1}\right) + \left(\frac{1}{a_1}-\frac{1}{b_1}\right)(\mu_1\sin 2\theta - \psi_2\cos 2\theta)\right] + \sigma\mu_1'. \tag{8}$$

The initial data are

$$\mu_i(0) = 1, \qquad \psi_i(0) = 0, \qquad \mu_i'(0) = \alpha_i,$$
$$\psi_i'(0) = \beta_i \qquad (i = 1, 2).$$

The quantities a_1, b_1, N, and θ are expressed in the usual manner in terms of μ_1, μ_2, ψ_1, and ψ_2.

If $H_0 = 0$ ($\sigma = 0$) and $\psi_i'(0) = 0$ at t = 0, we can obtain the solution in explicit form; here $\psi_1 = \psi_2 = \theta = 0$, while (8) gives

$$\mu_1'' = \frac{\beta}{\mu_1+\beta\mu_2}, \qquad \mu_2'' = \frac{1}{\mu_1+\beta\mu_2}. \tag{9}$$

The solution may be put as

$$\mu_1 = \beta\mu_2 + (\alpha_1 - \beta\alpha_2)\,\tau + (1-\beta),$$

$$\tau = \int_{1+\beta}^{z}\frac{dz}{\sqrt{(\alpha_1+\beta\alpha_2)^2 + 4\beta\ln[z/(1+\beta)]}},$$

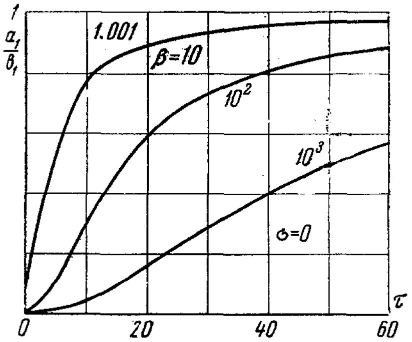

Fig. 2

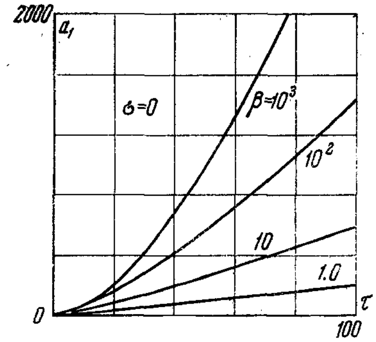

Fig. 3

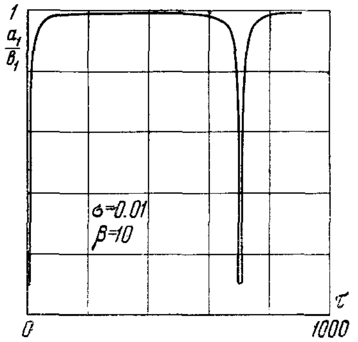

Fig. 4

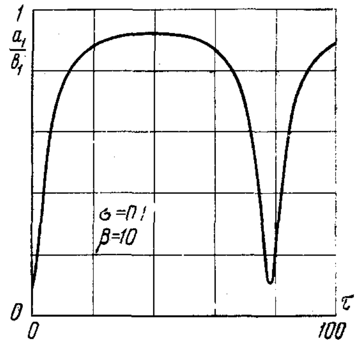

Fig. 5

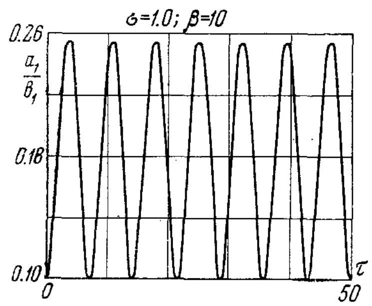

Fig. 6

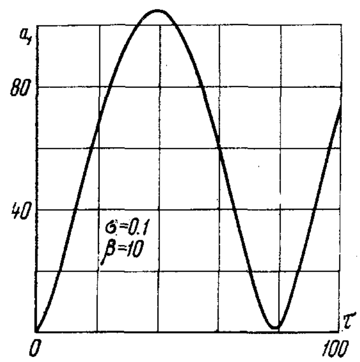

Fig. 7

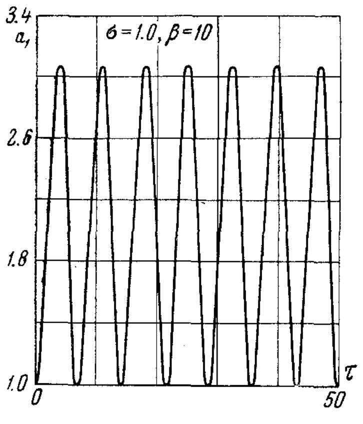

Fig. 8

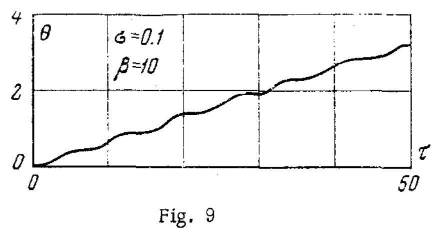

Fig. 9

$$z = 2\beta\mu_2 + (\alpha_1 - \beta\alpha_2)\tau + (1 - \beta). \quad (10)$$

The ratio of the semiaxes for the changing ellipse is

$$\frac{a_1}{b_1} = \frac{\beta\mu_2 + (\alpha_1 - \beta\alpha_2)\tau + (1 - \beta)}{\beta\mu_2}. \quad (11)$$

It follows from (10) that the ellipse, no matter what its initial shape, becomes a circle ($a_1/b_1 \to 1$) as $\tau \to \infty$. Figures 2 and 3 show the behavior of a_1/b_1 and $\alpha_i = 0$ and for several β. The result may be used to describe the behavior of a particle beam if it is reasonable to assume planar sections; an elliptical beam thus tends to become circular as it propagates. System (8) has been integrated numerically for $\sigma \neq 0$ and $\alpha_i = \beta_i = 0$; Figs. 4–9 show the results for $\beta = 10$ and various σ. The ratio of the semiaxes becomes periodic when $\sigma \neq 0$, as do the semiaxes themselves; but the precise character of the oscillations is very much dependent on σ, the amplitude and period being inversely related to σ. The orientation of the principal axes also changes, being rotated through θ relative to the fixed system, as shown in Fig. 9.

This implies that the beam has a periodic structure, with the principal axes rotating along the beam.

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